SPRINGER NATURE

Beiträge zur Algebra und Geometrie/ Contributions to Algebra and Geometry

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Reprinted from Beiträge zur Algebra und Geometrie/Contributions to Algebra and Geometry
Vol. 66, No. 1, pp. 135-157, March 2025

Beitr Algebra Geom (2025) 66:135–157
https://doi.org/10.1007/s13366-023-00728-5

ORIGINAL PAPER

Bisector fields of quadrilaterals

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Received: 7 July 2023 / Accepted: 26 November 2023 / Published online: 31 January 2024

Abstract
A bisector field is a maximal set $\mathbb{B}$ of paired lines in the plane such that each line in each pair crosses the other pairs in $\mathbb{B}$ in pairs of points that all share the same midpoint. We showed in a previous article that bisector fields are precisely the sets of pairs of lines that occur as asymptotes of hyperbolas from a pencil of affine conics, along with pairs of parallel lines arising from degenerate parabolas in the pencil. In this article we give a different application, this time to complete quadrilaterals and their nine-point conics. We show that every complete quadrilateral generates a bisector field as the set of bisectors of the quadrilateral paired according to an orthogonality condition in a geometry determined by the quadrilateral. The nine-point conic, so named because it passes through nine distinguished points of the quadrilateral, is shown to be the locus of midpoints of the bisector field associated to the quadrilateral, thus giving an interpretation of the other points on the nine-point conic. Our approach is analytic, and our results hold over any field of characteristic other than 2.

Keywords Quadrilateral · Bisector field · Nine-point conic · Pencil of conics

Mathematics Subject Classification Primary 51A20 · 51N10

1 Introduction

Throughout the paper $\Bbbk$ denotes a field of characteristic other than 2. Thus the results in this article hold not only in the Euclidean setting in which $\Bbbk$ is the field of real numbers but also, for example, in Galois geometries where $\Bbbk$ is a finite field. To our

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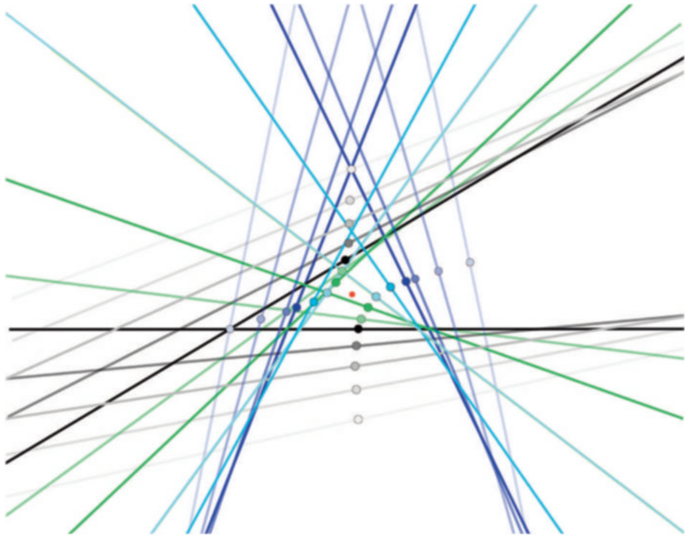

Fig. 1 Selected pairs from a bisector field, with each pair indicated by color. The marked points trace out a hyperbola and are the midpoints of bisectors of the same color. Each bisector bisects all other pairs with respect to the same midpoint

knowledge, our results are new for any choice of field $\mathbb{k}$. In Olberding and Walker (2024a), the notion of a bisector field was introduced as a collection $\mathbb{B}$ of pairs of lines in the affine plane such that (i) each line in each pair crosses the others pairs in $\mathbb{B}$ in pairs of points that all share the same midpoint, and (ii) $\mathbb{B}$ is not properly contained in a larger set of paired lines that satisfies (i); see Fig. 1.

It is not obvious that bisector fields exist, regardless of the choice of underlying field $\mathbb{k}$, but a natural source of examples is found in Olberding and Walker (2024a), where it is shown that bisector fields are precisely the sets of pairs of lines that arise from pencils of affine conics either as asymptotes of hyperbolas in the pencil or as pairs of parallel lines sharing a midline with degenerate parabolas in the pencil.

In this article we exhibit another natural source of examples of bisector fields, in this case complete quadrilaterals in the plane. To each quadrilateral Q we associate a nondegenerate symmetric bilinear form that defines a geometry on the plane in which the pairs of opposite sides of the quadrilateral Q are orthogonal. We then show that Q generates a bisector field whose line pairs, which include the pairs of opposite sides of Q, are all orthogonal in the geometry of the quadrilateral. The lines in this bisector field of the quadrilateral Q are simple to locate: They are the lines that bisect the quadrilateral, meaning they cross the pairs of opposite sides of the quadrilateral in pairs of points that share the same midpoint. However, the fact that these lines, the bisectors of the quadrilateral, form a bisector field is trickier and more surprising because it requires a symmetry that is *a priori* missing, namely that every orthogonal pair of bisectors of a quadrilateral Q is bisected by every bisector of Q. Thus not only does a bisector bisect the quadrilateral, the pair to which the bisector belongs is in return bisected by all the other bisectors of the quadrilateral. This can be seen in Fig. 1 by choosing any two pairs of lines of the same color and observing that all the other visible lines bisect the quadrilateral formed by these two pairs. Hence the bisectors of

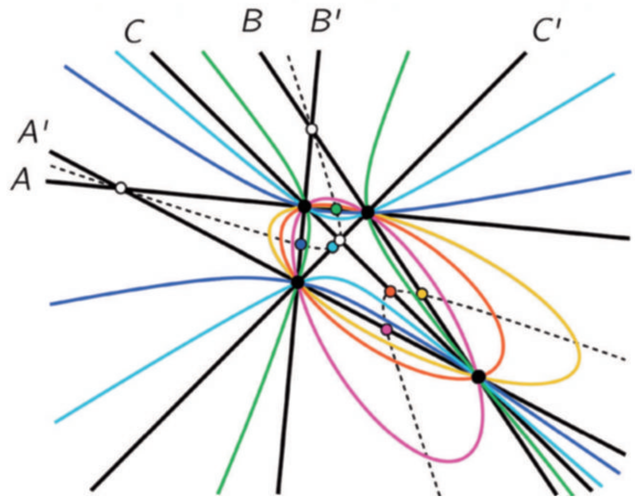

Fig. 2 These nine conics (3 ellipses, 3 nondegenerate hyperbolas and 3 degenerate hyperbolas) belong to the pencil of conics through the four black points. The nine conics are centered on the midpoints and diagonal points of the quadrilateral whose vertices are the four black points, and hence these centers are the nine distinguished points that define the nine-point conic for this quadrilateral, which is the dotted hyperbola. The asymptotes of the hyperbolas in the pencil here are also bisector pairs in the bisector field of the quadrilateral

a quadrilateral form a self-bisecting arrangement of lines in which the quadrilateral itself holds no privileged position.

We also prove that the midpoints of the bisectors of the quadrilateral Q all lie on a conic, possibly degenerate. When this conic is nondegenerate, all of the midpoints are, in the geometry of Q, the same "squared distance" away from the centroid of the quadrilateral. This conic that is comprised of bisector midpoints, the *bisector locus* of the bisector field defined by Q, is shown to coincide with a well-studied invariant of the quadrilateral, its *nine-point conic*, so called because it passes through nine canonical points of the quadrilateral, as in Fig. 2.

This conic, which appears in the work of Steiner and Beltrami and was rediscovered by British geometers in the late nineteenth century (see Vaccaro (2020)), is also the locus of centers of the conics in the pencil of conics through the vertices of the quadrilateral, as in Fig. 2. Thus, the notion of a bisector field helps complete the nineteenth-century picture of the nine-point conic by giving another interpretation of the other points on the nine-point conic as the midpoints of the bisectors of the otherwise "hidden" bisector field in which the quadrilateral resides.

Bisector fields are the subject of Olberding and Walker (2024a), and will be studied further in Olberding and Walker (2024b) and. (Except for results in Section 6, which use several results from Olberding and Walker (2024a), the present article is independent from these other articles.) In Olberding and Walker (2024b), we classify bisector fields up to affine equivalence. The classification introduces some subtleties that, unlike the results in the present paper, depend on the choice of field. For example, it is shown that over the field of real numbers there are exactly four bisector fields up to affine equivalence. In a future paper we will show that every bisector field gives rise to a field of conics, one at each point of the plane. Antipodal points on these conics are distinguished by the fact that they lie on pairs in the bisector field. In case $\Bbbk = \mathbb{R}$ and the conics are ellipses (which happens if and only if the nine-point conic is an ellipse), the field of ellipses is very closely related to a field of polarization ellipses in the theory of optics.

Although it has no bearing on the methods or results of the present paper, we mention that the present paper grew out of our interest in planar arrangement theorems involving rectangles inscribed in quadrilaterals; see for example (Olberding and Walker 2021, 2022, 2023; Schwartz 2020a, b; Tupan 2024). In the cited articles, various aspects of the flow of inscribed rectangles through complete quadrilaterals are studied. Affine transformations of these objects result in flows of inscribed parallelograms through complete quadrilaterals. We were led to the notion of a bisector in considering the extremal notion of degenerate parallelograms inscribed in complete quadrilaterals. When such parallelograms are extended to lines they are precisely the bisectors of the quadrilaterals.

Notation and terminology. By a (*complete*) *quadrilateral* $Q = ABA'B'$ we mean an arrangement of four distinct lines A, B, A', B', called the *sides of* Q, and the six points of intersections of these four lines. We require that not all four lines go through a single point. Also, adjacent sides of Q are not allowed to be parallel but opposite sides are. (Here we are being more restrictive in the definition of quadrilateral than in Olberding and Walker (2024a), where adjacent sides are allowed to be parallel.) The intersections of adjacent sides are the *vertices* of Q. The two lines through nonadjacent vertices are the *diagonals* of Q. Because of some technical considerations later in the paper and in Olberding and Walker (2024b), we deviate from conventional terminology and allow a quadrilateral to have three sides that pass through a single point, and so two vertices of Q (but at most two) can coincide. In this case we say the quadrilateral Q is *improper*, and otherwise Q is *proper*. The *centroid* of a quadrilateral Q is the midpoint of the midpoints of the diagonals of Q. Equivalently, the centroid of Q is the midpoint of the midpoints of any pair of opposite sides of Q. By a *parallelogram* we mean a quadrilateral whose pairs of opposite sides are parallel.

If a quadrilateral is proper, its four distinct vertices define a quadrangle in the traditional sense: A *quadrangle* $\mathcal{Q}$ consists of four distinct points in the plane, no three of which are collinear, the *vertices* of $\mathcal{Q}$, and the six lines through them, the *sides* of $\mathcal{Q}$. A quadrilateral Q *belongs* to a quadrangle $\mathcal{Q}$ if the vertices of Q are the same as those of $\mathcal{Q}$ and hence the four sides of the quadrilateral Q are among the six lines through the vertices of $\mathcal{Q}$. There are three quadrilaterals that belong to a quadrangle, and each of these is necessarily proper. The improper quadrilaterals are precisely the quadrilaterals that do not belong to a quadrangle. The centroid of a quadrangle $\mathcal{Q}$ is the (shared) centroid of any quadrilateral belonging to $\mathcal{Q}$.

We occasionally need to work in projective space. We denote the projective closure of n-dimensional affine space $\mathbb{A}^n(\Bbbk) = \Bbbk^n$ by $\mathbb{P}^n(\Bbbk)$. The points in $\mathbb{P}^n(\Bbbk)$ are written in homogeneous coordinates $[x_1 : x_2 : \cdots : x_{n+1}]$, where $x_1, x_2, \ldots, x_{n+1} \in \Bbbk$ are not all zero and this point is the line in $\Bbbk^{n+1}$ that passes through the origin and the point $(x_1, x_2, \ldots, x_{n+1})$. We identify $\mathbb{A}^n(\Bbbk)$ with the set of points in $\mathbb{P}^n(\Bbbk)$ of the form $[x_1 : \cdots : x_n : 1]$. We will only need $\mathbb{P}^1(\Bbbk)$ and $\mathbb{P}^2(\Bbbk)$ in this article. By *the line at infinity* in $\mathbb{P}^2(\Bbbk)$ we mean the line $\{[x_1 : x_2 : 0] : x_1, x_2 \in \Bbbk \text{ not both zero}\}$.

Throughout the paper, we use the following convention.

If $tX - uY + v = 0$ defines a line L in the plane $\Bbbk \times \Bbbk$, then we can assume without loss of generality that if $u = 0$ then $t = 1$, and if $u \neq 0$ then $u = 1$. With this assumption, we have a canonical choice of coefficients for the line, and so we may unambiguously write t_L for t, u_L for u and v_L for v.

2 *Q*-orthogonal pairs

For a quadrilateral Q, we introduce an inner product on $\Bbbk \oplus \Bbbk$ under which the pairs of opposite sides and diagonals of Q are orthogonal. In the next definition we are using the notational convention from the last section.

Definition 2.1 Let $Q = ABA'B'$ be a quadrilateral, possibly improper, and let

$$\begin{aligned}\alpha &= t_A u_B u_{A'} u_{B'} - u_A t_B u_{A'} u_{B'} + u_A u_B t_{A'} u_{B'} - u_A u_B u_{A'} t_{B'}\\ \beta &= t_A u_B t_{A'} u_{B'} - u_A t_B u_{A'} t_{B'}\\ \gamma &= t_A t_B t_{A'} u_{B'} - t_A t_B u_{A'} t_{B'} + t_A u_B t_{A'} t_{B'} - u_A t_B t_{A'} t_{B'}.\end{aligned}$$

Let Φ_Q be the quadratic form

$$\Phi_Q(X, Y) = \gamma X^2 - 2\beta XY + \alpha Y^2.$$

For the vector space $\Bbbk^2$, define an inner product (i.e., symmetric bilinear form) by

$$\langle \mathbf{v}, \mathbf{w} \rangle_Q = \mathbf{v}^T \begin{bmatrix} \gamma & -\beta \\ -\beta & \alpha \end{bmatrix} \mathbf{w} \ \text{ for all } \mathbf{v}, \mathbf{w} \in \Bbbk^2.$$

Therefore, $\Phi_Q(\mathbf{v}) = \langle \mathbf{v}, \mathbf{v} \rangle_Q$ for all $\mathbf{v} \in \Bbbk^2$.

Different quadrilaterals can induce the same inner product since these objects depend only on the slopes of the sides of the quadrilateral, not their position.

Lemma 2.2 *The inner product $\langle -, - \rangle_Q$ is nondegenerate.*

Proof It is straightforward to verify

$$\beta^2 - \alpha\gamma = (t_A u_B - t_B u_A)(t_B u_{A'} - t_{A'} u_B)(t_{A'} u_{B'} - t_{B'} u_{A'})(t_{B'} u_A - t_A u_{B'}). \quad (1)$$

Since adjacent sides of Q are not parallel, $\beta^2 - \alpha\gamma \neq 0$. The determinant of the matrix in Definition 2.1 is $\alpha\gamma - \beta^2$, so this matrix is invertible, and hence the inner product is nondegenerate. □

Definition 2.3 Two lines ℓ_1, ℓ_2 in $\Bbbk^2$ are *Q-orthogonal* if $\langle (u_{\ell_1}, t_{\ell_1}), (u_{\ell_2}, t_{\ell_2}) \rangle_Q = 0$.

To prove the main result of this section, Proposition 2.5, we recall that an *involution* (or more precisely, an involutive homography) on a line ℓ in the projective plane $\mathbb{P}^2(\Bbbk)$ is a projective transformation Λ from ℓ to ℓ for which $\Lambda \circ \Lambda$ is the identity map on ℓ. Two points p and q on ℓ are *conjugate* under Λ if $\Lambda(p) = q$.

Lemma 2.4 (Desargues' Involution Theorem) *Let $\mathscr{Q}$ be a quadrangle in $\mathbb{P}^2(\Bbbk)$, and let ℓ be a line in $\mathbb{P}^2(\Bbbk)$ that does not pass through a vertex of $\mathscr{Q}$. There is an involution Λ on ℓ such that if p and q are the points of intersection of ℓ and a conic through the vertices of $\mathscr{Q}$, then p and q are conjugate under Λ. Moreover, any two conjugate pairs of points on ℓ determine the involution.*

For a proof of Lemma 2.4 in our setting of arbitrary fields of characteristic other than 2, see [Berger (1987), 14.2.8.3, p. 125] or Nguyen (2020). For a proof using bisector methods, see [Olberding and Walker (2024a), Corollary 5.6].

Proposition 2.5 *If Q is a quadrilateral, possibly improper, then the pairs of opposite sides of Q are Q-orthogonal, as are the diagonals of Q.*

Proof Write $Q = ABA'B'$, and let Λ_Q denote the projective transformation on the line at infinity given for each point $[x : y : 0]$ by

$$\Lambda_Q([x : y : 0]) = [\beta x - \alpha y : \gamma x - \beta y : 0].$$

Since $\beta^2 - \alpha\gamma \neq 0$ by Lemma 2.2, Λ_Q is an involution on the line at infinity. A straightforward calculation using the expressions for α, β, γ in Definition 2.1 shows the points at infinity $[u_A : t_A : 0]$ and $[u_{A'} : t_{A'} : 0]$ for A and A' are conjugate under Λ_Q, as are the points at infinity for B and B'. Comparing the definition of Λ_Q with that of the inner product in Definition 2.1, it follows that two lines in $\Bbbk^2$ are Q-orthogonal if and only if their points at infinity are conjugate under Λ_Q.

It remains to consider the diagonals of Q. If Q is an improper quadrilateral, the diagonals of Q are opposite sides of Q and so the diagonals are Q-orthogonal. Assume Q is a proper quadrilateral. Let $\mathscr{Q}$ be the quadrangle to which Q belongs. By Lemma 2.4 the quadrangle $\mathscr{Q}$ defines an involution Λ on the line at infinity such that the points at infinity of any conic through the vertices of $\mathscr{Q}$ are conjugate, and this involution is uniquely determined by any two conjugate pairs. Each pair of opposite sides of $\mathscr{Q}$ comprises a (degenerate) conic passing through the vertices of $\mathscr{Q}$, and so the points at infinity for opposite sides of $\mathscr{Q}$ are conjugate under Λ. Therefore, since the points at infinity for A and A' are conjugate under Λ and Λ_Q, as are the points at infinity for B and B', we have $\Lambda = \Lambda_Q$ by Lemma 2.4. Consequently, the diagonals of Q, as opposite sides of $\mathscr{Q}$, have points at infinity that are conjugate under Λ_Q. Hence the diagonals of Q are Q-orthogonal. □

Remark 2.6 With Λ_Q as in the proof of Proposition 2.5, the points $[x : y : 0]$ on the line at infinity that are conjugate to themselves correspond to the null vectors for Φ_Q, that is,

$$\Lambda_Q([x : y : 0]) = [x : y : 0] \iff \Phi_Q(x, y) = 0.$$

Thus a line is Q-orthogonal to itself if and only if its point at infinity is conjugate to itself under Λ_Q.

Corollary 2.7 *If Q and Q' are proper quadrilaterals sharing the same vertices, then two lines are Q-orthogonal if and only if they are Q'-orthogonal.*

Proof The proof of Proposition 2.5 shows that $\Lambda_Q = \Lambda_{Q'}$ since the points at infinity for the pairs of opposite sides and diagonals of Q are conjugate under both Λ_Q and $\Lambda_{Q'}$. Also, as noted in the proof of Proposition 2.5, two lines are Q-orthogonal if and only if their points at infinity are conjugate under Λ_Q. The analogous statement holds for $\Lambda_{Q'}$, and so the corollary follows. □

The next proposition will be needed later when using affine transformations to reduce to simpler cases.

Proposition 2.8 *Let Q be a quadrilateral, and let f be an invertible linear transformation of the plane. There is $\lambda \in \mathbb{k}$ such that for all $\mathbf{v}, \mathbf{w} \in \mathbb{k}^2$,*

$$\langle \mathbf{v}, \mathbf{w}\rangle_Q = \lambda \langle f(\mathbf{v}), f(\mathbf{w})\rangle_{f(Q)}.$$

Thus a pair of lines ℓ_1, ℓ_2 is Q-orthogonal if and only if $f(\ell_1), f(\ell_2)$ is $f(Q)$-orthogonal.

Proof Write $Q = ABA'B'$, and let $a, b, c, d \in \mathbb{k}$ such that

$$f(x, y) = (ax + by, cx + dy)$$

for all $x, y \in \mathbb{k}$. Define a map f_∞ on the line at infinity by

$$f_\infty([x : y : 0]) = [ax + by : cx + dy : 0]$$

for all points $[x : y : 0]$ on the line at infinity. Let Λ_Q be the involution from the proof of Proposition 2.5. Then $f_\infty \Lambda_Q f_\infty^{-1}$ is an involution on the line at infinity for which

$$[u_{f(A)} : t_{f(A)} : 0], [u_{f(A')} : t_{f(A')} : 0] \text{ and } [u_{f(B)} : t_{f(B)} : 0], [u_{f(B')} : t_{f(B')} : 0]$$

are conjugate pairs. Since these pairs are pairs of points at infinity of sides of the quadrilateral $f(Q)$, they are conjugate under $\Lambda_{f(Q)}$, and so Lemma 2.4 implies $\Lambda_{f(Q)} = f_\infty \Lambda_Q f_\infty^{-1}$. Let x, y be in the algebraic closure $\overline{\mathbb{k}}$ of $\mathbb{k}$, where at least one of x, y is nonzero. Using Remark 2.6, $\Phi_{f(Q)}(f(x, y)) = 0$ if and only if $\Phi_Q(x, y) = 0$. Thus $\Phi_{f(Q)} \circ f$ and Φ_Q have the same zeroes in $\mathbb{P}^1(\overline{\mathbb{k}})$, and so there is λ in the algebraic closure of $\mathbb{k}$ such that $\lambda \Phi_Q = \Phi_{f(Q)} \circ f$. Moreover, $\lambda \in \mathbb{k}$ since the coefficients of these two polynomials are in $\mathbb{k}$. The proposition now follows from the parallelogram law for inner products. □

3 Basic properties of bisectors

Bisectors of collections of conics are defined in Olberding and Walker (2024a), and we may apply this definition to our setting. A line ℓ *crosses a pair of lines* $\mathsf{P} = \{\ell_1, \ell_2\}$ if it is distinct from both ℓ_1 and ℓ_2 and is not parallel to both of them. In this case, we write $\mathrm{mid}_{\mathsf{P}}(\ell)$ for the midpoint of the two points where ℓ meets P. At most one of these two points may be at infinity, and if one is at infinity we define $\mathrm{mid}_{\mathsf{P}}(\ell)$ to be the point at infinity for ℓ. If the midpoint is not at infinity, we say it is finite.

Definition 3.1 Let ℓ be a line.

(1) If $\mathscr{P}$ is a collection of pairs of lines, then ℓ *bisects* $\mathscr{P}$ if $\text{mid}_{\mathsf{P}}(\ell) = \text{mid}_{\mathsf{P}'}(\ell)$ for all pairs $\mathsf{P}, \mathsf{P}' \in \mathscr{P}$ that ℓ crosses. This common midpoint is the *midpoint* of the bisector ℓ.
(2) If Q is a quadrilateral or quadrangle, then ℓ *bisects* Q if ℓ bisects the pairs of opposite sides of Q. We write $\text{mid}_Q(\ell)$ for the midpoint of the bisector ℓ.

When referring to a bisector ℓ of a quadrilateral Q, we mean the line and its midpoint $\text{mid}_Q(\ell)$. This point is determined by any two points where the bisector crosses a pair of opposite sides of Q that are not parallel to ℓ. (There must be at least one such pair of opposite sides since the vertices of Q are, by definition, not at infinity.)

Proposition 3.2 *Let Q be a quadrilateral.*

(1) *The midpoint* $\text{mid}_Q(\ell)$ *is finite for all bisectors ℓ of Q.*
(2) *A line ℓ passing through a vertex of Q bisects Q if and only if it is a side of Q or a diagonal of Q.*
(3) *Two distinct parallel lines are bisectors of Q if and only if these lines are parallel to a pair of sides or diagonals of Q.*
(4) *All lines parallel to a pair of parallel sides or diagonals of Q bisect Q.*

Proof Statement (1) follows from the fact that adjacent sides of a quadrilateral (as defined in the introduction) cannot be parallel. Statement (2) is a simple argument that depends on the fact that any bisector that goes through one vertex of a quadrilateral must go through another. For (3) and (4), use the fact that if P is a pair of lines, and ℓ and ℓ' are distinct parallel lines not parallel to either line in P, then for every line ℓ'' parallel to ℓ and ℓ', the line through $\text{mid}_{\mathsf{P}}(\ell)$ and $\text{mid}_{\mathsf{P}}(\ell')$ goes through $\text{mid}_{\mathsf{P}}(\ell'')$. This fact implies that if ℓ and ℓ' are bisectors of Q, then every line parallel to ℓ and ℓ' is a bisector of Q. Thus there are bisectors through the vertices of Q parallel to ℓ and ℓ', and by (2), these bisectors must be sides or diagonals of Q. Statements (3) and (4) follow from these considerations. □

Lemma 3.3 *Let $\mathscr{Q}$ be a quadrangle. The bisectors of $\mathscr{Q}$ that pass through a vertex of $\mathscr{Q}$ are the sides of $\mathscr{Q}$. The bisectors that do not pass through a vertex are the lines whose involutions induced by $\mathscr{Q}$ as in Lemma 2.4 are reflections.*

Proof The first assertion follows from Proposition 3.2(2). To prove the second, suppose a line ℓ bisects $\mathscr{Q}$ but does not pass through a vertex of $\mathscr{Q}$. The involution on ℓ given by Lemma 2.4 is determined by where ℓ meets pairs of opposite sides of $\mathscr{Q}$ (which is possibly at infinity), and so ℓ is a bisector of $\mathscr{Q}$ if and only if this involution is a reflection. □

By definition, bisection of quadrangles requires bisection of all three pairs of opposite sides. Proposition 3.4 shows it is enough to check bisection for two pairs of opposite sides.

Proposition 3.4 *If Q and Q' are proper quadrilaterals sharing the same vertices, then the bisectors of Q and Q' are the same. Also, $\Phi_Q = \lambda\Phi_{Q'}$ for some $\lambda \in \Bbbk$, and two lines are Q-orthogonal if and only if they are Q'-orthogonal.*

Proof By assumption, Q and Q' belong to the same quadrangle $\mathscr{Q}$. To prove the first assertion, it suffices to show that if ℓ is a line that bisects a quadrilateral Q, then ℓ bisects the pairs of opposite sides of $\mathscr{Q}$. Let ℓ be such a line. If ℓ passes through a vertex of $\mathscr{Q}$, then ℓ is a side of $\mathscr{Q}$ by Proposition 3.2, and hence ℓ is a bisector of $\mathscr{Q}$. If instead ℓ does not pass through a vertex of $\mathscr{Q}$, then by Lemma 2.4 the involution on ℓ induced by $\mathscr{Q}$ is determined by the points where the two pairs of opposite sides of Q meet ℓ. Since ℓ bisects these pairs and an involution on a line is determined by any two pairs of conjugate points, this involution is a reflection and so ℓ bisects $\mathscr{Q}$ by Lemma 3.3.

To prove the rest of the proposition, note that by equation (1) from the proof of Lemma 2.2, the discriminant of Φ_Q is nonzero, so the quadratic form Φ_Q has two distinct zeroes in $\mathbb{P}^1(\overline{\mathbb{k}})$, where $\overline{\mathbb{k}}$ is the algebraic closure of $\mathbb{k}$. Similarly, $\Phi_{Q'}$ has two distinct zeroes. By Corollary 2.7, two lines are Q-orthogonal if and only if they are Q'-orthogonal, so it follows from Remark 2.6 that Φ_Q and $\Phi_{Q'}$ have the same zeroes in $\mathbb{P}^1(\overline{\mathbb{k}})$. Therefore, $\Phi_Q = \lambda\Phi_{Q'}$ for some $\lambda \in \overline{\mathbb{k}}$. In fact, $\lambda \in \mathbb{k}$ since the coefficients of Φ_Q and $\Phi_{Q'}$ are in $\mathbb{k}$. □

The next proposition, which will be used often in what follows, shows that two bisectors can share the same midpoint only in a very special case.

Proposition 3.5 *Let Q be a quadrilateral. Two distinct bisectors of Q share the same midpoint if and only if the vertices of Q are the vertices of a parallelogram. In this case, the shared midpoint is the centroid of Q.*

Proof We will use the following observation, which can be checked with an easy calculation:

(†) If P is a pair of non-parallel lines and p is a point in the plane that is not the intersection of the lines in P, then there is a unique line ℓ passing through p such that $p = \mathrm{mid}_{\mathsf{P}}(\ell)$.

Write $Q = ABA'B'$, and suppose first that Q is an improper quadrilateral. Then the vertices of Q are not the vertices of a parallelogram. Also, three sides, say A, A', B, meet at a point. If p is a point not at this intersection, then by (†) there is at most one bisector whose midpoint is p. Otherwise, suppose p is the point of intersection of A, A' and B.

If there are two bisectors with midpoint p, then these two bisectors are sides or diagonals of Q by Proposition 3.2(2) and hence must be among A, A' and B. Thus at least one of A or A' has midpoint p, say A, and so A bisects B, B' with midpoint p. Since B passes through p, this implies B' also passes through p, a contradiction to the fact that not all four sides of Q go through a point. Thus if Q is an improper quadrilateral, the vertices of Q are not the vertices of a parallelogram and no two bisectors share a midpoint.

Now suppose Q is a proper quadrilateral. If the vertices of Q are not the vertices of a parallelogram, then there is a quadrilateral Q' that has the same vertices as Q such that no sides of Q' are parallel. Thus if p is a point in the plane, there is a pair of non-parallel opposite sides of Q' such that p is not the intersection of these two lines, and so by (†), there is at most one bisector having p as its midpoint. Otherwise, if the vertices of Q are the vertices of a parallelogram P, then any line through the center p of P is a bisector of P having midpoint p. Since Q and P belong to the same quadrangle, Proposition 3.4 implies these lines through p are bisectors of Q having midpoint p.

Finally, if ℓ and ℓ' are different bisectors of Q with the same midpoint p, then, as we have shown, the vertices of Q are the vertices of a parallelogram P and the centroid of Q is the center of P. By Proposition 3.4, ℓ and ℓ' bisect the diagonals of the parallelogram P, and so by (†), since ℓ and ℓ' are distinct, p must lie on the intersection of the diagonals, and hence p is the center of P, which is the centroid of Q. □

Proposition 3.5 is not true for the more general quadrilaterals considered in Olberding and Walker (2024a), those quadrilaterals that can have a vertex at infinity. See [Olberding and Walker (2024a), Figure 2(c)].

4 Quadrilaterals in standard form

For use in later sections, we give an equational description of all the bisectors of a quadrilateral Q in what we call standard form, a simpler case we can always reduce to via an affine transformation if Q is not a parallelogram.

Definition 4.1 A quadrilateral $Q = ABA'B'$ is in *standard form* if A is the line $Y = 0$ and A' is the line $X = 0$.

If Q is a quadrilateral that is not a parallelogram, then Q has a pair of opposite sides, say A, A', that are not parallel, and so there is an affine transformation that carries A and A' onto the axes of $\Bbbk^2$. Thus every quadrilateral that is not a parallelogram can be transformed into a quadrilateral in standard form. We use this often in the next two sections.

If the quadrilateral $Q = ABA'B'$ is in standard form, then neither of the lines B or B' is parallel to the X or Y-axis, and so (with our notational convention from the Introduction) $u_B = u_{B'} = 1$, which implies the slopes of the lines B and B' are t_B and $t_{B'}$, respectively. The product $t_B t_{B'}$ of these slopes is a fundamental invariant of Q.

Definition 4.2 The *coefficient* of a quadrilateral $Q = ABA'B'$ in standard form is the product $\mu = t_B t_{B'}$ of the slopes of B and B'.

The coefficient is never 0 since the slopes of B and B' are not 0. In standard form, Q-orthogonality is easy to detect in terms of the coefficient of Q:

Lemma 4.3 *If a quadrilateral Q is in standard form with coefficient μ, then*

$$\Phi_Q(X, Y) = Y^2 - \mu X^2.$$

Moreover, two lines ℓ and ℓ' not parallel to the X or Y-axes are Q-orthogonal if and only if the product of their slopes is μ.

Proof Write $Q = ABA'B'$, where the sides of Q are as in Definition 4.1. Substituting the data for A, B, A' and B' into the definitions of α, β and γ in Definition 2.1, we obtain $\alpha = 1$, $\beta = 0$, $\gamma = -t_B t_{B'} = -\mu$, and so $\Phi_Q(X, Y) = \gamma X^2 - 2\beta XY + \alpha Y^2 = Y^2 - \mu X^2$. Also, since $u_\ell = u_{\ell'} = 1$, the lines ℓ and ℓ' are Q-orthogonal if and only if $t_\ell t_{\ell'} - \mu = 0$. □

The next proposition gives an equational description of all of the bisectors of a quadrilateral in standard form.

Proposition 4.4 *Let Q be a quadrilateral in standard form with centroid (h, k) and coefficient μ.*

(1) *A line ℓ is a bisector of Q with midpoint $(p, q) \neq (0, 0)$ if and only if*

$$q(q - 2k) - \mu p(p - 2h) = 0 \text{ and } \ell \text{ has equation } qX + pY - 2pq = 0.$$

(2) *If $(h, k) = (0, 0)$, every line through $(0, 0)$ is a bisector of Q with midpoint $(0, 0)$.*

(3) *If $(h, k) \neq (0, 0)$, then the unique bisector whose midpoint is the line $kY + \mu hX = 0$.*

Proof (1) Write $Q = ABA'B'$. Since A is the line $Y = 0$ and A' the line $X = 0$, the unique line ℓ passing through (p, q) that intersects A and A' at points that have (p, q) as their midpoint is the line passing through $(0, 2q)$ and $(2p, 0)$. This line ℓ has equation $qX + pY - 2pq = 0$, and if ℓ is distinct from A and A', the line ℓ is the unique bisector of A, A' with midpoint (p, q). Thus it suffices to show:

($\star$) The line $\ell : qX + pY - 2pq = 0$ bisects Q with midpoint (p, q) if and only if

$$q(q - 2k) - \mu p(p - 2h) = 0.$$

We first calculate the centroid of Q. Since Q is in standard form, the equations for B and B' are given by $Y = t_B X + v_B$ and $Y = t_{B'} X + v_{B'}$, with $t_B t_{B'} = \mu$. The vertices of the quadrilateral Q are

$$\begin{aligned} A \cdot B &= \left(-\frac{t_{B'} v_B}{\mu}, 0\right), \quad B \cdot A' = (0, v_B), \quad A' \cdot B' = (0, v_{B'}), \\ B' \cdot A &= \left(-\frac{t_B v_{B'}}{\mu}, 0\right). \end{aligned} \tag{2}$$

Therefore, since (h, k) is the mean of the four vertices of Q,

$$-4h\mu = t_{B'} v_B + t_B v_{B'} \text{ and } 4k = v_B + v_{B'}. \tag{3}$$

Now we verify ($\star$). Suppose first ℓ is not parallel to B or B'. Then $pt_B + q \neq 0$ and $pt_{B'} + q \neq 0$, and so the intersections of ℓ with B and B' are the points

$$\ell \cdot B = \left(\frac{p(2q - v_B)}{pt_B + q}, \frac{q(2pt_B + v_B)}{pt_B + q}\right) \quad \ell \cdot B' = \left(\frac{p(2q - v_{B'})}{pt_{B'} + q}, \frac{q(2pt_{B'} + v_{B'})}{pt_{B'} + q}\right)$$

The line ℓ bisects the pair B, B' with midpoint (p, q) if and only if (p, q) is the midpoint of $\ell \cdot B$ and $\ell \cdot B'$. Using this observation, the expressions for h and k in (3), and the fact that $t_B t_{B'} = \mu$ and $(p, q) \neq (0, 0)$, a midpoint calculation shows ℓ bisects the pair B, B' with midpoint (p, q) if and only if $q(q-2k) - \mu p(p-2h) = 0$. This verifies the proposition in the case in which ℓ is not parallel to B or B'.

Now suppose ℓ is parallel to B or B'. We give the argument for the case in which ℓ is parallel to B; the case in which ℓ is parallel to B' is similar. Since by assumption ℓ is parallel to B, and B is not parallel to $X = 0$ or $Y = 0$, it follows that $p \neq 0$ and $q \neq 0$. The slope of ℓ is $-q/p$, so $q = -pt_B$. Using this fact, the assumption that $t_B t_{B'} = \mu$ and the expressions for h and k from (3), we obtain

$$q(q-2k) - \mu p(p-2h) = \frac{p}{2}(t_B - t_{B'})(2pt_B + v_B).$$

Since $p \neq 0$,

$$q(q-2k) - \mu p(p-2h) = 0 \iff t_B = t_{B'} \text{ or } 2pt_B + v_B = 0.$$

To complete the proof of ($\star$), it suffices to show (still under the assumption that ℓ is parallel to B) that the line $qX + pY - 2pq = 0$ bisects Q with midpoint (p, q) if and only if $t_B = t_{B'}$ or $2pt_B + v_B = 0$.

If $\ell = B$, then since ℓ goes through the point $(2p, 0)$ and B has equation $Y = t_B X + v_B$, it follows that $2pt_B + v_B = 0$. If instead $\ell \neq B$ and ℓ bisects B, B' with midpoint (p, q), then since ℓ is a bisector distinct from B and parallel to B, Proposition 3.2(3) implies B is parallel to B' so that $t_B = t_{B'}$.

Conversely, still under the assumption that ℓ is parallel to B, suppose $t_B = t_{B'}$ or $2pt_B + v_B = 0$. In the former case, B is parallel to B' and so since ℓ is parallel to B and B', $\text{mid}_Q(\ell)$ is the midpoint of $\ell \cdot A$ and $\ell \cdot A'$, and hence ℓ bisects Q with midpoint (p, q). In the case in which $2pt_B + v_B = 0$, the parallel lines ℓ and B share the point $(2p, 0)$ and hence are equal. Thus ℓ bisects B, B' with midpoint (p, q). This completes the proof of ($\star$).

(2) Since Q is in standard form and all four sides do not go through the same point, it cannot be that $v_B = v_{B'} = 0$. Thus if $(h, k) = (0, 0)$, then equations (3) imply $v_B = -v_{B'}$ and $t_B = t_{B'}$, which in turn implies B and B' are parallel lines with $(0, 0)$ lying on their midline[1], and so every line through the origin bisects the pair B, B' with the origin as its midpoint. Since the origin is the intersection of A and A', every line through the origin bisects Q with the origin as its midpoint.

(3) Suppose $(h, k) \neq (0, 0)$. We show the line ℓ defined by $kY + \mu hX = 0$ bisects Q with midpoint $(0, 0)$. Once this is proved, the fact that ℓ is the unique bisector with midpoint $(0, 0)$ follows from Proposition 3.5. Note that the line ℓ bisects A, A' with midpoint $(0, 0)$ since A and A' meet at the origin. Thus it remains to show ℓ bisects the pair B, B' with midpoint $(0, 0)$. If B is parallel to B', then using the expressions for h and k from equations (3), we have $-4\mu h = t_B v_{B'} + t_{B'} v_B = 4t_B k$. Thus $-\mu h = t_B k$, proving that ℓ is parallel to B and B'. In this case, ℓ bisects Q with midpoint $(0, 0)$ because ℓ bisects A, A' with midpoint $(0, 0)$ and is parallel to B and B', and so the claim is proved.

[1] The *midline* of a pair of parallel lines L and L' is the line consisting of the midpoints of the points on L and the points on L', i.e., it is the line midway between the two parallel lines.

Otherwise, B is not parallel to B'. If ℓ is B or B', then ℓ clearly bisects Q, so we assume ℓ is neither of these lines. By Proposition 3.2(3), ℓ is not parallel to B or B' since these last two lines are not parallel to each other. Therefore, $\mu h + t_B k \neq 0$ and $\mu h + t_{B'} k \neq 0$, and so ℓ intersects B and B' in the points

$$\ell \cdot B = \left(-\frac{k v_B}{\mu h + t_B k}, \frac{v_B \mu h}{\mu h + t_B k} \right), \quad \ell \cdot B' = \left(-\frac{k v_{B'}}{\mu h + t_{B'} k}, \frac{v_{B'} \mu h}{\mu h + t_{B'} k} \right).$$

Equations (3) imply (0, 0) is the midpoint of these two points, and so the line ℓ bisects B, B' with midpoint (0, 0). Thus the line $kY + \mu h X = 0$ bisects Q. □

Corollary 4.5 *If Q is a quadrilateral in standard form, then the centroid of Q is the origin if and only if the vertices of Q are the vertices of a parallelogram.*

Proof If the vertices of Q are the vertices of a parallelogram P, then since Q is in standard form and A and A' are not parallel, it follows that A and A' are the diagonals of P and hence the origin (the intersection of A and A') is the center of P, which is the centroid of Q since P and Q share the same vertices. The converse follows from Propositions 3.5 and 4.4(2). □

The last corollary shows one of the main advantages of reducing to standard form.

Corollary 4.6 *If two quadrilaterals in standard form share the same coefficient and the same centroid, then both quadrilaterals have the same bisectors.*

Proof This follows from Proposition 4.4 since the description of the bisectors in this proposition depends only on the centroid and coefficient of the quadrilateral. □

The converse of the corollary is also true. This is proved in Olberding and Walker (2024b) using different methods.

5 The bisector locus

In this section we describe the *bisector locus* of a quadrilateral Q (or quadrangle $\mathscr{Q}$), which is the locus of midpoints of the bisectors of Q (or $\mathscr{Q}$). By Proposition 3.4, the bisector locus of a quadrangle is the bisector locus of any quadrilateral belonging to the quadrangle. For the sake of stating the next theorem, we adapt the notion of a diagonal point of a quadrangle to that of a quadrilateral: A *diagonal point* of a quadrilateral Q is a point that is the intersection of a pair of opposite sides of Q or the diagonals of Q. Since the quadratic form Φ_Q can be viewed as a squared norm for the inner product space defined in Definition 2.1 and determined by Q, the idea behind the next theorem is that the bisector locus is either a circle or a pair of lines in the geometry of Q.

Theorem 5.1 *Let Q be a quadrilateral, let (h, k) be the centroid of Q and let (a, b) be a diagonal point of Q that is not at infinity. The bisector locus of Q is given by*

$$\Phi_Q(X - h, Y - k) = \Phi_Q(a - h, b - k).$$

Thus the bisector locus is a conic with center (h, k).

Proof Suppose first Q is in standard form and $(a, b) = (0, 0)$. By Lemma 4.3, $\Phi_Q(X, Y) = Y^2 - \mu X^2$, and by Proposition 4.4, the bisector locus is the zero set of the polynomial $Y^2 - 2kY - \mu(X^2 - 2hX)$. Completing the square, the bisector locus is defined by the equation $(Y - k)^2 - \mu(X - h)^2 = k^2 - \mu h^2$. Thus $\Phi_Q(X - h, Y - k) = \Phi_Q(h, k)$, which proves the theorem when Q is in standard form and (a, b) is the origin.

Now suppose $Q = ABA'B'$ is not necessarily in standard form. Using Proposition 3.4 we can switch quadrilaterals if necessary and assume that A and A' are not parallel and intersect at the point $\mathbf{a} = (a, b)$. Thus there is an affine transformation f that carries $\mathbf{a}$ to the origin and carries Q onto a quadrilateral in standard form. Let g be a linear transformation such that $f(\mathbf{x}) = g(\mathbf{x}) - g(\mathbf{a})$ for all $\mathbf{x} \in \Bbbk^2$. Let $\mathbf{c} = (h, k)$ be the centroid of Q. Since affine transformations preserve midpoints and hence bisectors, a point $\mathbf{x} = (x, y) \in \Bbbk^2$ is on the bisector locus of Q if and only if $f(\mathbf{x})$ is a point on the bisector locus of $f(Q)$. Let μ be the coefficient of the quadrilateral $f(Q)$. Since $\Phi_{f(Q)}$ depends only on the slopes of the lines that comprise $f(Q)$, it follows that $\Phi_{f(Q)} = \Phi_{g(Q)}$. The centroid of $f(Q)$ is $f(\mathbf{c})$ and the theorem has been verified for quadrilaterals in standard form, so we use Proposition 2.8 to obtain that for each $\mathbf{x}$ in $\Bbbk^2$,

$$\begin{aligned} f(\mathbf{x}) \in \text{bisector locus} &\iff \Phi_{f(Q)}(f(\mathbf{x}) - f(\mathbf{c})) = \Phi_{f(Q)}(f(\mathbf{a}) - f(\mathbf{c})) \\ &\iff \Phi_{g(Q)}(g(\mathbf{x} - \mathbf{c})) = \Phi_{g(Q)}(g(\mathbf{a} - \mathbf{c})) \\ &\iff \Phi_Q(\mathbf{x} - \mathbf{c}) = \Phi_Q(\mathbf{a} - \mathbf{c}). \end{aligned}$$

This proves the theorem. □

A conic over $\Bbbk$ is *degenerate* if the polynomial that defines it is a product of linear polynomials over the algebraic closure of $\Bbbk$. Theorem 5.2 gives a criterion for when the bisector locus is degenerate.

Theorem 5.2 *The bisector locus of a quadrilateral Q is degenerate if and only if there is a pair L, L' of parallel sides or diagonals of Q. In this case, the bisector locus of Q is the union of the midline of L and L' and the line through the midpoints of L and L'.*

Proof Assume first $Q = ABA'B'$ is in standard form. By Lemma 4.3, $\Phi_Q(X, Y) = Y^2 - \mu X^2$. Suppose the bisector locus is degenerate. If $(h, k) = (0, 0)$, then by Corollary 4.5, Q has a pair of parallel sides or diagonals, so we may assume $(h, k) \neq (0, 0)$. By Theorem 5.1, the fact that the locus is degenerate implies $\Phi_Q(h, k) = k^2 - \mu h^2 = 0$, so by Proposition 4.4(1), for every $t \in \Bbbk$, since

$$tk(tk - 2k) - \mu th(th - 2h) = 0,$$

we have that (th, tk) is the midpoint of a bisector given by $tkX + thY - 2t^2hk = 0$. Each such bisector is distinct (since $(h, k) \neq (0, 0)$) and has the same slope, so by Proposition 3.2(3), Q has a pair of parallel sides or diagonals.

Conversely, still assuming Q is in standard form, suppose there is a pair of sides or diagonals of Q that are parallel. By Proposition 3.4 we can switch quadrilaterals if necessary in order to assume B and B' are parallel and Q is in standard form. Let μ be the coefficient of Q. Since $t_B = t_{B'}$ and $t_B t_{B'} = \mu$, we have $t_B^2 = \mu$. Also, equation (3) from the proof of Proposition 4.4(1) implies $-4\,h\mu = t_B(v_B + v_{B'}) = 4t_B k$, and so $k = -t_B h$. Lemma 4.3 and Theorem 5.1 imply then that the bisector locus of Q is the zero set of

$$Y(2k - Y) - \mu X(2h - X) = (t_B X + Y)(t_B X - Y + 2k).$$

The line $Y = -t_B X$ is the line through the midpoints of the parallel sides B and B' and the line $Y = t_B X + 2k$ is the midline of B and B', so the theorem is proved if Q is in standard form.

Now suppose $Q = ABA'B'$ is not necessarily in standard form. By Proposition 3.4, we can assume A and A' are not parallel and hence meet at a point (a, b). Let $\mathbf{c} = (h, k)$ and $\mathbf{a} = (a, b)$. By Theorem 5.1, the bisector locus of Q is degenerate if and only if $\Phi_Q(\mathbf{a} - \mathbf{c}) = 0$. Let g be a linear transformation such that the map f from the plane to itself defined by $f(\mathbf{x}) = g(\mathbf{x}) - g(\mathbf{a})$ for all $\mathbf{x} \in \mathbb{k}^2$ is an affine transformation that carries Q onto a quadrilateral in standard form and $\mathbf{a}$ to the origin. Since $f(Q)$ is in standard form, $f(Q)$, and hence Q, has a pair of parallel sides or diagonals if and only if $\Phi_{f(Q)}(f(\mathbf{a}) - f(\mathbf{c})) = 0$. Since $\Phi_{f(Q)}$ depends only on the slopes of the lines that comprise $f(Q)$, the quadrilateral Q has a pair of parallel sides or diagonals if and only if $\Phi_{g(Q)}(g(\mathbf{a} - \mathbf{c})) = 0$; if and only if $\Phi_Q(\mathbf{a} - \mathbf{c}) = 0$, where this last equivalence follows from Proposition 2.8. □

The *nine-point conic* of a quadrangle $\mathcal{Q}$ is the unique conic passing through the six midpoints of the quadrangle, i.e., the midpoints of each pair of vertices, and the three diagonal points of $\mathcal{Q}$, the points at which opposite sides meet, where possibly some of these diagonal points are at infinity. For a proof of the existence of the nine-point conic over an arbitrary field of characteristic $\neq 2$, see [Berger (1987), 16.5.5.1, p. 198]. In our case, the existence follows from the fact that the bisector locus is a conic that passes through these same nine points.

Corollary 5.3 *The bisector locus of a quadrangle $\mathcal{Q}$ is the conic through the diagonal points and midpoints of $\mathcal{Q}$ and hence is the nine-point conic for $\mathcal{Q}$.*

Proof The six sides of $\mathcal{Q}$ are bisectors with respect to their midpoints, and so the midpoints of these sides lie on the bisector locus. We claim the diagonal points of $\mathcal{Q}$ are also midpoints of bisectors of $\mathcal{Q}$. If a side A of $\mathcal{Q}$ is parallel to its opposite side A', then the diagonal point for A and A' is the point at infinity for these lines. In this case, Theorem 5.2 implies the midline of A and A' is part of the bisector locus. Since the point at infinity for the midline is the diagonal point of A and A', this point is a point at infinity for the bisector locus of $\mathcal{Q}$.

Now suppose A is not parallel to A'. If the other two pairs of opposite sides of $\mathcal{Q}$ form a parallelogram, then A and A' are the diagonals of this parallelogram and these lines meet at the center of the parallelogram, in which case the diagonal point $A \cdot A'$ is the side midpoint of both A and A' and hence the midpoint of a bisector. Otherwise, if there is another pair B, B' of opposite sides that are not parallel, then $A \cdot A'$ is the midpoint of a point on B and a point on B', and hence the line through these two points is a bisector of $ABA'B'$ having $A \cdot A'$ as its midpoint. By Proposition 3.4, this line bisects $\mathscr{Q}$ and so $A \cdot A'$ is on the bisector locus of $\mathscr{Q}$. □

6 The bisector field of a quadrilateral

By an affine conic over $\Bbbk$, we mean the zero set of a quadratic polynomial in $\Bbbk[X, Y]$ (quadratic, for short). A hyperbola is a centered conic that has two distinct points at infinity. Its asymptotes are the two lines through the center of the hyperbola that meet the hyperbola at infinity. A parabola is a conic with one point at infinity and an ellipse a conic with no points at infinity. A pair of parallel lines or a double line is a degenerate parabola.

A pair of lines in the affine plane is a *degeneration* of an affine conic defined by a quadratic f if there is $\lambda \in \Bbbk$ such that the union of these two lines is the set of zeroes of $f + \lambda$. Neither an ellipse nor a parabola has a degeneration, and a hyperbola has a unique degeneration, namely its asymptotes. A pair of parallel lines ℓ, ℓ' has as its degenerations the pairs of lines parallel to ℓ and ℓ' that share the same midline as the pair ℓ, ℓ'. See [Olberding and Walker (2024a), Proposition 2.2] for more details.

To each quadrilateral $Q = ABA'B'$, we associate a pencil of affine conics. Let f and g be degenerate conics whose zero sets are $A \cup A'$ and $B \cup B'$, respectively. The *pencil of* Q, denoted $\text{Pen}(Q)$, is the collection of affine conics defined by the nonzero polynomials in

$$\Bbbk f_1 + \Bbbk f_2 \; = \; \{\alpha f_1 + \beta f_2 : \alpha, \beta \in \Bbbk\}$$

Every nonzero polynomial in $\Bbbk f_1 + \Bbbk f_2$ is a quadratic, since otherwise the degree two homogeneous parts of f_1 and f_2 are scalar multiples of each other, and hence the pairs of lines defined by f_1 and f_2, i.e., the pairs of opposite sides of Q, meet at infinity, contrary to the fact that Q is a quadrilateral. If Q is a proper quadrilateral, then the pencil of Q is simply the set of conics passing through all four vertices of Q. If instead Q is an improper quadrilateral and p is the double vertex of Q, the vertex through which three sides pass, say A, A', B, then the pencil of Q is the set of conics that pass through all three vertices and are tangent to B at p.

The *asymptotic pencil* of Q is the set of reducible conics of the form $f + \lambda$, where $\lambda \in \Bbbk$ and $f \in \Bbbk f_1 + \Bbbk f_2$; i.e., the asymptotic pencil of Q is the set of conics whose zero sets are the degenerations of the conics in $\text{Pen}(Q)$. It consists of the asymptotes of the hyperbolas in $\text{Pen}(Q)$ and the pairs of parallel lines that share their midline with that of a pair of parallel lines in $\Bbbk f_1 + \Bbbk f_2$, if any such parallel lines exist. See Fig. 3 for an illustration. The asymptotic pencil is the subject of Olberding and Walker (2024a).

The next lemma, which is crucial in the proofs of several of the results that follow, gives a first connection between bisectors and asymptotic pencils.

Lemma 6.1 *[Olberding and Walker (2024a), Lemma 5.2], A line ℓ is a bisector of a quadrilateral Q if and only if it is part of a degeneration of a conic in the pencil of Q.*

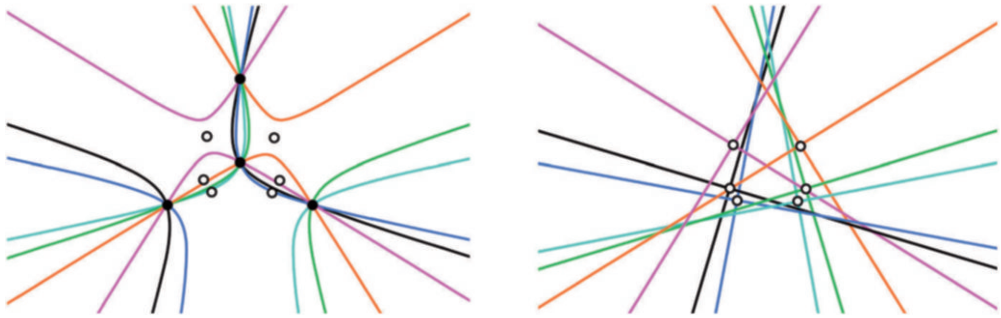

Fig. 3 On the left are several conics belonging to the pencil through the four black points. The centers of these conics are marked with white points. On the right are the asymptotes of these same conics. The asymptotes of the black hyperbola at left are the black lines at right. By Theorem 6.4, these pairs belong to the bisector field of the quadrilateral defined by the pencil. The midpoints of the bisectors are the white points

Thus if ℓ is a bisector of a quadrilateral Q, there is another bisector ℓ' of Q such that ℓ, ℓ' is a degeneration of a conic in $\text{Pen}(Q)$. This suggests a natural pairing on the bisectors of Q, one that proves essential in the context of bisector fields, which are considered at the end of this section. This pairing will turn out to be a combination of Q-orthogonality and another pairing involving antipodal points (see Theorem 6.4). Because of some pathologies involving parallel lines, Q-orthogonality itself is not quite sufficient to handle all cases, so we combine this relationship with an antipodal one across the centroid of Q.

Definition 6.2 A pair of bisectors of the quadrilateral Q is Q-*antipodal* if the midpoint of the midpoints of the bisectors is the centroid of Q. The pair is a Q-*pair* if it is Q-antipodal and Q-orthogonal.

We will show in Corollary 6.5 that in most cases, a pair of bisectors is Q-antipodal if and only if it is Q-orthogonal. This is not true in all cases, however, as the following example shows.

Example 6.3 The bisector locus of a parallelogram Q is a pair of lines meeting at the center of Q and parallel to the sides of Q, as in Fig. 4. It is straightforward to verify that the Q-pairs of bisectors of Q are (a) the pairs of lines parallel to a pair of sides of Q and having the same midline as these sides, (b) the midlines of the parallel pairs in (a), where each midline is paired with itself, and (c) the diagonals of the parallelograms whose pairs of opposite sides are the pairs in (a). Where the lines in (a) cross a midline from (b) are the midpoints of these bisectors, while the bisectors in (b) and (c) all have the centroid of Q as their midpoint. Any pair of lines parallel to a midline in (b) is a Q-orthogonal pair of bisectors of Q but need not be a Q-antipodal pair. Similarly, any pair of lines through the center of Q is a Q-antipodal pair of bisectors that need not be Q-orthogonal.

The next theorem gives a quadrilateral-specific way to detect the degenerations of the conics in the pencil of a quadrilateral that is independent of reference to the pencil itself.

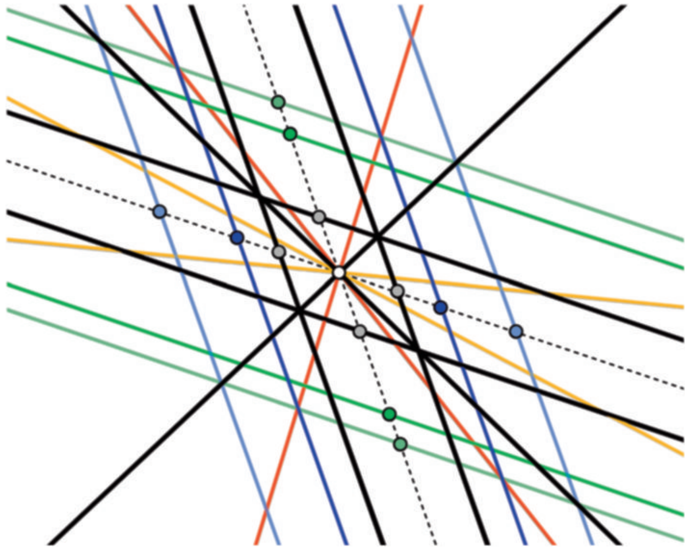

Fig. 4 Pairs from a bisector field of a parallelogram. All the lines through the center point are bisectors having the center as their midpoint. The lines through the center are paired according to the shade of gray with which they are colored. The parallel lines are paired via antipodal midpoints

Theorem 6.4 *The Q-pairs of bisectors of Q are precisely the degenerations of the conics in the pencil of Q.*

Proof By Proposition 3.4, if Q is a parallelogram we may switch to a quadrilateral that has the same vertices as Q but for which Q is not a parallelogram. Thus, after an affine transformation, we may assume $Q = ABA'B'$ is in standard form. Let $v = v_B v_{B'}$, let (h, k) be the centroid of Q and let $\mu = t_B t_{B'}$ be the coefficient of Q. Using equations (2) and (3) from the proof of Proposition 4.4(1), along with the fact that $t_B t_{B'} = \mu$, it follows that the conics in the pencil of Q are given by the equations of the form

$$\mu a X^2 + \mu b XY + aY^2 - 4a\mu hX - 4akY + av = 0. \tag{4}$$

where a and b range over the elements in $\Bbbk$ with a and b not both 0.

We first prove that if ℓ_1, ℓ_2 is a degeneration of one of the conics of the form (4), and at least one of ℓ_1, ℓ_2 is parallel to the X or Y-axis, then ℓ_1, ℓ_2 is a Q-pair of bisectors of Q. Let ℓ_1, ℓ_2 be a pair of lines that is a degeneration of a conic in (4). By Lemma 6.1, ℓ_1 and ℓ_2 are bisectors of Q. Suppose first that one of these lines is parallel to the Y-axis. Then there are $\lambda, a, b, t, u, v, w \in \Bbbk$ such that at least one of a, b is nonzero, at least one of t, u is nonzero and

$$\mu a X^2 + \mu b XY + aY^2 - 4a\mu hX - 4akY + av + \lambda = (X + w)(tX - uY + v).$$

This implies $a = 0$ and hence $t = v = w = 0$, so that ℓ_1 and ℓ_2 are the X and Y-axes. Thus ℓ_1, ℓ_2 is a Q-pair of bisectors since the axes are a pair of opposite sides of Q. A similar argument shows that if one of ℓ_1 and ℓ_2 is parallel to the X-axis, then $a = 0$ in equation (4), and ℓ_1, ℓ_2 are the axes and hence a Q-pair of bisectors of Q.

We can therefore reduce to the case that $a = 1$ in equation (4). The next thing to prove is that if ℓ_1, ℓ_2 is a degeneration of a conic of the form

$$f_b(X, Y) := \mu X^2 + \mu bXY + Y^2 - 4\mu hX - 4kY + v, \text{ where } b \in \Bbbk,$$

such that neither ℓ_1 nor ℓ_2 is parallel to an axis, then ℓ_1, ℓ_2 is a Q-pair of bisectors of Q. Let ℓ_1, ℓ_2 be a degeneration of f_b for some $b \in \Bbbk$, where neither ℓ_1 nor ℓ_2 is parallel to an axis. Then ℓ_1 and ℓ_2 are bisectors of Q by Lemma 6.1, and there are $\lambda, t_1, v_1, t_2, v_2 \in \Bbbk$ such that

$$f_b(X, Y) + \lambda = (t_1X - Y + v_1)(t_2X - Y + v_2).$$

Equating coefficients yields

$$\mu = t_1t_2 \quad -4\mu h = t_1v_2 + t_2v_1 \quad 4k = v_1 + v_2.$$

Since $\mu = t_1t_2$, the pair ℓ_1, ℓ_2 is Q-orthogonal. To see that ℓ_1, ℓ_2 is a Q-antipodal pair, consider the quadrilateral $Q' = A\ell_1A'\ell_2$. Its vertices are $(0, v_1)$, $(0, v_2)$, $(-v_1/t_1, 0)$, $(-v_2/t_2, 0)$. The centroid of Q', which is the mean of these vertices, is

$$\left(\frac{-v_1t_2 - v_2t_1}{4t_1t_2}, \frac{v_1 + v_2}{4}\right) = \left(\frac{-v_1t_2 - v_2t_1}{4\mu}, \frac{v_1 + v_2}{4}\right) = (h, k).$$

The midpoints of ℓ_1 and ℓ_2 as bisectors of Q are the same as the midpoints of ℓ_1 and ℓ_2 as sides of Q', so since the midpoint of the midpoints of a pair of opposite sides of a quadrilateral is the centroid of that quadrilateral, we conclude that ℓ_1, ℓ_2 is a Q-antipodal pair. In all cases, we have proved that if ℓ_1, ℓ_2 is a degeneration of a conic in the pencil of Q, then ℓ_1, ℓ_2 is a Q-pair of bisectors of Q.

Conversely, suppose ℓ_1, ℓ_2 is a Q-pair of bisectors of Q. We show ℓ_1, ℓ_2 is a degeneration of a conic in the pencil of Q. Lemma 6.1 implies that since ℓ_1 is a bisector of Q, there is a line ℓ_2' such that ℓ_1, ℓ_2' is a degeneration of a conic in the pencil of Q. By what we have established, ℓ_1, ℓ_2' is therefore a Q-pair of bisectors. Thus ℓ_1 is Q-orthogonal to ℓ_2 and ℓ_2', and so ℓ_2 and ℓ_2' are parallel. Also, since ℓ_1, ℓ_2 and ℓ_1, ℓ_2' are Q-antipodal pairs, ℓ_2 and ℓ_2' share the same midpoint, so $\ell_2 = \ell_2'$. Therefore, ℓ_1, ℓ_2 is a degeneration of a conic in the pencil of Q. □

Theorem 6.4 implies that the Q-pairs of bisectors of Q that are not parallel to each other are precisely the asymptotes of the hyperbolas in the pencil of Q. The point of the next corollary is that for almost all choices of quadrilaterals and pairs of bisectors, the properties of being Q-antipodal and Q-orthogonal are redundant.

Corollary 6.5 *Let Q be a quadrilateral.*

(1) *Every Q-orthogonal pair of bisectors not both parallel to two sides or diagonals of Q is Q-antipodal and hence is a Q-pair.*
(2) *Every Q-antipodal pair of bisectors that do not share a midpoint is Q-orthogonal and hence is a Q-pair.*

Proof (1) Suppose ℓ_1, ℓ_2 is a Q-orthogonal pair of bisectors of Q that are not parallel to a pair of sides or diagonals of Q. By Lemma 6.1, there is a bisector ℓ_2' such that ℓ_1, ℓ_2' is a degeneration of a conic in the pencil of Q. By Theorem 6.4, ℓ_1, ℓ_2' is a Q-pair. Since ℓ_1, ℓ_2 is a Q-orthogonal pair of bisectors, ℓ_2 and ℓ_2' are parallel. If $\ell_2 \neq \ell_2'$, then by Proposition 3.2(3), ℓ_2 is parallel to a pair of sides or diagonals of Q, contrary to assumption. Thus $\ell_2 = \ell_2'$, and so ℓ_1, ℓ_2 is a degeneration of a conic in the pencil of Q. By Theorem 6.4, ℓ_1, ℓ_2 is a Q-pair.

(2) Suppose ℓ_1, ℓ_2 is a Q-antipodal pair of bisectors that do not share a midpoint. As in the proof of (1), there are bisectors ℓ_1' and ℓ_2' such that ℓ_1, ℓ_2' and ℓ_1', ℓ_2 are Q-pairs of bisectors. Since these pairs are Q-antipodal, ℓ_2' and ℓ_2 share the same midpoint, as do ℓ_1 and ℓ_1'. By assumption, ℓ_1 and ℓ_2 do not share a midpoint, so by Proposition 3.5, either $\ell_1 = \ell_1'$ or $\ell_2 = \ell_2'$. As in the proof of (1), this implies ℓ_1, ℓ_2 is a Q-pair. □

Whether two bisectors are Q-orthogonal is *a priori* not easy to determine geometrically since orthogonality depends on the inner product defined in Section 2. The next corollary, however, shows that for a generically chosen quadrilateral, there is a simple geometric way to distinguish whether two bisectors are Q-orthogonal.

Corollary 6.6 *If the quadrilateral Q has no parallel sides or diagonals, then a pair of bisectors of Q is Q-orthogonal if and only if it is Q-antipodal.*

Proof Apply Proposition 3.5 and Corollary 6.5. □

The following corollary can be compared to the more general setting in Olberding and Walker (2024a), where quadrilaterals can have a vertex at infinity. In that case, bisectors may not have unique partners; see [Olberding and Walker (2024a), Lemma 6.2].

Corollary 6.7 *If ℓ is a bisector of a quadrilateral Q, then there is a unique bisector ℓ' for which ℓ, ℓ' is a Q-pair.*

Proof Let (p, q) be the midpoint of ℓ, and let (p', q') be the point that is antipodal to (p, q) across the centroid of Q. If $(p, q) \neq (p', q')$, then by Proposition 3.5 and the fact that the bisector locus is a centered conic (Theorem 5.1), there is a unique bisector ℓ' with midpoint (p', q'), and so by Corollary 6.5, ℓ, ℓ' is a Q-pair and ℓ' is the only bisector of Q with this property.

Otherwise, if $(p, q) = (p', q')$, then (p, q) is the centroid of Q since ℓ, ℓ' is a Q-antipodal pair. There is a unique line ℓ' through (p', q') that is Q-orthogonal to ℓ. If $\ell = \ell'$, then clearly ℓ' is a bisector; if $\ell \neq \ell'$, then ℓ' is a bisector with midpoint $(p, q) = (p', q')$ by Proposition 3.5 and the discussion in Example 6.3. The line ℓ' is the unique bisector of Q such that the pair ℓ, ℓ' is a Q-pair. □

In Olberding and Walker (2024a) it is shown that each line in an asymptotic pencil of a quadrilateral bisects every pair of lines in the asymptotic pencil with respect to the same midpoint. (In fact, each such line also bisects the conics in the pencil of Q [Olberding and Walker (2024a), Theorem 5.3].) This is expressed using the notion of a bisector field, the definition of which we recall now.

Definition 6.8 A collection $\mathbb{B}$ of pairs of lines is a *bisector arrangement* if each line in each pair in $\mathbb{B}$ bisects $\mathbb{B}$. A bisector arrangement is *trivial* if all lines in the arrangement meet at the same point (possibly at infinity) or every pair is a translation of every other pair. A *bisector field* is a nontrivial bisector arrangement that cannot be extended to a larger bisector arrangement. The bisector field is *affine* if no line in one pair is parallel to a line in another pair.

The reason for "affine" in the definition of affine bisector field is that these are the bisector fields for which no bisector in the bisector field has a midpoint at infinity. In Olberding and Walker (2024a), a more general definition of quadrilateral than the present one is allowed, one in which adjacent sides can be parallel. Say that a arrangement of four lines is an *infinite quadrilateral* if it is a quadrilateral as defined in the Introduction, except that two adjacent sides are parallel. (This quadrilateral is "infinite" because it has a vertex at infinity.) One of the main results of Olberding and Walker (2024a) is that a collection of a paired lines is a nontrivial asymptotic pencil if and only if it is a bisector field; see [Olberding and Walker (2024a), Theorem 6.3]. In this case the asymptotic pencil is generated by a quadrilateral or an infinite quadrilateral (Olberding and Walker 2021, Proposition 4.5). We use this to deduce the next theorem for our slightly more restrictive setting.

Theorem 6.9 *A collection $\mathbb{B}$ of paired lines is an affine bisector field if and only if these line pairs are the Q-pairs of bisectors of a quadrilateral Q. In particular, every bisector of Q bisects every Q-pair of bisectors of Q.*

Proof Suppose $\mathbb{B}$ is an affine bisector field. By [Olberding and Walker (2024a), Theorem 6.3], $\mathbb{B}$ is the set of degenerations of conics in a pencil of an *a priori* possibly infinite quadrilateral Q. Since the pairs of opposite sides of Q are in $\mathbb{B}$ and $\mathbb{B}$ is affine, Q is a quadrilateral as defined in the Introduction since in this case adjacent sides of Q cannot be parallel. By Theorem 6.4, $\mathbb{B}$ is the set of Q-pairs of bisectors of Q.

Conversely, suppose $\mathbb{B}$ is the set of Q-pairs of bisectors of Q. Then $\mathbb{B}$ is the set of degenerations of the conics in the pencil of Q by Theorem 6.4. By [Olberding and Walker (2024a), Theorem 6.3], $\mathbb{B}$ is a bisector field. If $\mathbb{B}$ is not affine, then $\mathbb{B}$ contains two pairs A, A' and B, B' with A parallel to B but not parallel to B'. The midpoint of A as a bisector of the pair B, B' is therefore at infinity, and since $\mathbb{B}$ is a bisector field, this implies A is parallel to at least one line in every pair in $\mathbb{B}$. But then Q, as a quadrilateral in $\mathbb{B}$, must have a pair of adjacent sides that are parallel to A, contrary to the assumption that Q is a (finite) quadrilateral. Therefore, $\mathbb{B}$ is affine. □

By [Olberding and Walker (2024a) Theorem 5.3], each bisector of Q also bisects every conic in the pencil of Q. When Q is a proper quadrilateral, this assertion can also be derived from Desargues' Involution Theorem (Lemma 2.4).

The collection of Q-pairs of bisectors of a quadrilateral Q is the *bisector field of* Q.

Corollary 6.10 *Two quadrilaterals have the same bisectors if and only if they have the same bisector fields.*

Proof Suppose Q_1 and Q_2 are quadrilaterals that have the same bisectors. We show Q_1 and Q_2 have the same bisector fields. It suffices by Theorem 6.9 to show that if

ℓ, ℓ' is a Q_1-pair of bisectors of Q_1, then ℓ, ℓ' is Q_2-pair. Since Q_1 and Q_2 have the same bisectors, they have the same bisector locus and hence the same centroid. Thus a pair ℓ, ℓ' of bisectors of Q_1 is Q_1-antipodal if and only if it is Q_2-antipodal. If the midpoints of ℓ and ℓ' are distinct, then by Corollary 6.5 this implies ℓ, ℓ' is a Q_1-pair if and only if it is a Q_2-pair.

Otherwise, suppose ℓ, ℓ' is a Q_1-pair of bisectors of Q_1 with the same midpoint. Since these bisectors are Q_1-antipodal, this shared midpoint is the centroid of Q_1. By Corollary 6.7, there is a bisector ℓ'' of Q_2 such that ℓ, ℓ'' is a Q_2-pair, and so since ℓ, ℓ'' is Q_2-antipodal and the centroid of Q_1 is the centroid of Q_2, it follows that ℓ and ℓ'' have the centroid as a shared midpoint. If $\ell' = \ell''$, the claim is proved, so suppose $\ell' \neq \ell''$. By Proposition 3.5, the vertices of Q_1 are the vertices of a parallelogram, and similarly for Q_2, since in both cases there are two distinct bisectors with the same midpoint. The description in Example 6.3 implies that since Q_1 and Q_2 have the same bisector locus, ℓ, ℓ' is a Q_1-pair if and only if ℓ, ℓ' is a Q_2-pair. This proves that Q_1 and Q_2 have the same bisectors if and only if they have the same bisector fields. □

Corollary 6.11 *The only pairing that makes the bisectors of a quadrilateral Q a bisector field is Q-pairing.*

Proof Apply Theorem 6.9 and Corollary 6.10. □

In the paper Olberding and Walker (2024b) we classify bisector fields in terms of envelopes of lines tangent to curves.

Acknowledgements We thank the referees for helpful comments that improved the paper.

Funding No financial support was received for this manuscript.

Data availibility statement Not applicable.

Declarations

Conflict of interest The authors declare that there is no conflict of interest in this work.

References

Berger, M.: Geometry II. Springer, Berlin (1987)

Nguyen, N.P.: A generalization of Desargues' involution theorem. J. Geom. Graph. **24**(1), 65–72 (2020)

Olberding, B., Walker, E.: The conic geometry of rectangles inscribed in lines. Proc. Am. Math. Soc. **149**, 2625–2638 (2021)

Olberding, B., Walker, E.: Paths of rectangles inscribed in lines over fields. Beitr. Algebra Geom. **64**, 55–80 (2023)

Olberding, B., Walker, E.: Rectangles conformally inscribed in lines. J. Geom. **113**, 9 (2022)

Olberding, B., Walker, E.: Bisector fields and pencils of conics, Aequ. Math. (2024a). https://doi.org/10.48550/arXiv.2305.08324

Olberding, B., Walker, E.: Bisector fields and projective duality (2024b). https://doi.org/10.48550/arXiv.2306.08612

Schwartz, R.: Four lines and a rectangle. Exp. Math. (2020). https://doi.org/10.1080/10586458.2019.1671923

Schwartz, R.: A trichotomy for rectangles inscribed in Jordan loops. Geom. Dedicata **208**, 177–196 (2020)

Tupan, A.: Poncelet closure for the hyperbola, preprint (2024)

Vaccaro, M.: Historical origins of the nine-point conic—the contribution of Eugenio Beltrami. Hist. Math. **51**, 26–48 (2020)